On the METHOD of taking RADICALS out of EQUATIONS. By MR. D. MOONEY, *A. B. Trinity College, Dublin.* Communicated by WHITLEY STOKES, *M. D. F. T. C. D. and M. R. I. A.*

AMONG the rules delivered by the incomparable Sir Isaac Newton for the Reduction of Equations, the following is a principal one: Read June 18th, 1796.

"Si cui surdæ quantitati irreducibili litera involvatur ad cujus dimensiones, æquatio ordinanda est, cæteri omnes termini ad contrarias partes, cum signis mutatis transferendi sunt, & utraque pars æquationis in se semel multiplicanda, si radix quadratica sit, vel bis si sit cubica."

IF the letter be involved in a single surd, the operation is easily performed by simple involution; but if in two or more surds, of the same or different dimensions, or if there be a number of radical quantities involved in the terms of the equation, the operation for clearing such an equation from radicals, is somewhat more difficult.

IN

In the 97th Section of his Analysis, Doctor Hales shews the method of taking quadratic surds out of an equation, provided the number of terms be not greater than four, but if there be a fifth term, whether rational or surd, he is of opinion that the equation cannot, by that method, be cleared from surds. Were this the case, we would have no other alternative, than to recur to the method of Monsieur Fermat, by feigning the surds equal to an assumed letter, and thence by means of as many simple equations, as there are surds, to take these letters out of the equation; but this is a work of so much labour, that it is sufficient to deter a person slightly acquainted with algebra. However, by considering the nature of the surds that arise after involution, it will readily appear that there is no such limit, nor any need of recurring to an operation so laborious.

Let the equation proposed consist of five quadratic surds, if these can be rendered rational, four quadratics and a rational may be reduced at least with the same ease,

$$\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d} + \sqrt{f},$$

it is plain, that any equation, consisting of five surds, may be reduced to this form, thence after involution will arise the fewest number of *surd* rectangles (Doctor Saunderson shews in his algebra that if two quantities be irrational, their product will be so too) for if the equation was divided into parts of one and four surds, the result would have six surd rectangles, and from the former position there will arise but four.

By

By ſquaring the ſides of the equation propoſed, we will have,

$$a+b+2\sqrt{ab}=c+d+f+2\sqrt{cd}+2\sqrt{cf}+2\sqrt{df}.$$

If all the combinations that can be made out of the trinomial $\sqrt{c}+\sqrt{d}+\sqrt{f}$ be obſerved, which are only three, we will ſee, that the ſuffixed quantities of every two of theſe admit one common diviſor; place then any two of theſe combinations at the ſame ſide, and the reſt of the terms at the other, and aſſuming the rational quantities collected $=p$ we have,

$$p+2\sqrt{ab}-2\sqrt{cd}=2\sqrt{cf}+2\sqrt{df}, \text{ and by ſquaring}$$

$$p^2+4ab+4cd+4p\sqrt{ab}-4p\sqrt{cd}-8\sqrt{abcd}=4cf+4df+8f\sqrt{dc}.$$

Of theſe ſurd rectangles, it will appear, that two coaleſce, i. e. have the ſame ſuffixed quantity, and the ſuffixed quantities of the others admit a common diviſor. By placing the rationals collected $\overline{p^2+4ab+4dc-4cf-4df}=q$, and the coefficient $8f-4p=r$, we have $4p\sqrt{ab}-8\sqrt{abcd}=r\sqrt{dc}-q$, and ſquaring,

$16p^2ab+64abcd-64pab\sqrt{cd}=r^2dc+q^2-2qr\sqrt{cd}$: now putting the rationals $=s$, and the coefficient, $64pab-2qr=t$, the equation will ſtand, $s=t\sqrt{cd}$, and ſquaring, $s^2=t^2cd$, which latter equation is free from ſurds. It is plain that aſſuming letters for the rationals, is only for brevity, and no way affects the ſurds. Thus it will appear, how an equation, involving five quadratic ſurds, is rendered rational, by the ſimple rules of involution and tranſpoſition.

Having

HAVING thus ſhewn, that this method has more latitude than has been generally imagined, I ſhall endeavour to evince the truth of it in the caſe of a ſexinomial; thereby demonſtrating its univerſality, and ſhewing how the ſurds, one by one, vaniſh, notwithſtanding the enormous appearance of ſurd rectangles at firſt; for the number of ſurd rectangles, ariſing from the involution of a multinomial ſurd quantity to the ſquare, is always equal to the ſum of the natural numbers between o and the number of the parts connected by the ſigns + or –, and therefore rapidly encreaſes.—Vide Saunderſon's Alg. Tom. 2, Section 422, (ſub finem;) & Simpſon's Alg. Chapter on Combinations.

LET there be propoſed the equation, conſiſting of a rational and five quadratic ſurds, $x + \sqrt{a} + \sqrt{b} - \sqrt{c} - \sqrt{d} - \sqrt{f} = 0$.

AFTER tranſpoſition, the equation muſt either have two parts at one ſide, and four at the other, in which caſe, after involution, there will be ſeven ſurd rectangles; or one part at one ſide, and five at the other, from whence would ariſe, after involution, ten ſurd rectangles; or finally, three parts at each ſide of the equation, from whence, after involution, reſult ſix ſurd rectangles, i. e. three at each ſide. Let us take the equation from whence the leaſt number of ſurd rectangles reſult;

$$x + \sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d} + \sqrt{f}$$

ſquared $\overline{x^2 + a + b + 2x\sqrt{a} + 2x\sqrt{b} + 2\sqrt{ab} = c + d + f + 2\sqrt{ca}}$ $+ 2\sqrt{cf} + 2\sqrt{df}.$

LET

Let the rationals collected $= g$, and transferring to the other ſide as well the rationals as the ſimple quadratic $2x\sqrt{b}$, the equation will be $2x\sqrt{a} + 2\sqrt{ab} = 2\sqrt{cd} + 2\sqrt{cf} + 2\sqrt{df} - 2x\sqrt{b} - g$.

For brevity ſake I ſhall hereafter aſſume letters for the rationals of the equations, and transfer each ſurd as it comes out to a ſimilar one wherever it is found, annexing the ſign according to the ſide it is placed at; ſquaring the above equation we have

$$\begin{array}{ll} h + 8xa\sqrt{b} & = i \mid 8c\sqrt{df} + 8d\sqrt{cf} - 8x\sqrt{bcd} - 4g\sqrt{cd} - 8x\sqrt{bcf} - 8x\sqrt{bdf} \\ -4gx\sqrt{b} & \quad -4g\sqrt{df} - 4g\sqrt{cf} \qquad\qquad +8f\sqrt{cd} \end{array}$$

Here then we have an equation free from the ſurd $\sqrt{a}$ and ſince the coefficients of the ſurds can by no means affect the poſſibility of clearing it, and we are ſo far only concerned, I ſhall omit them, and ſubſtituting k for the rationals $i - h$, the equation will be $\sqrt{b} + \sqrt{bcd} + \sqrt{bcf} + \sqrt{bdf} = \sqrt{df} + \sqrt{cf} + \sqrt{cd} - k$ and ſquaring.

$$\begin{array}{ll} l + 2b\sqrt{cd} + 2b\sqrt{cf} + 2b\sqrt{df} & = m + 2f\sqrt{cd} + 2d\sqrt{fc} - 2k\sqrt{df} \\ + 2bf\sqrt{cd} + 2bd\sqrt{cf} + 2bc\sqrt{df} & \quad - 2k\sqrt{cd} - 2k\sqrt{fc} + 2c\sqrt{df}, \end{array}$$

whence neglecting coefficients after uniting the ſurds; we have $\sqrt{cd} + \sqrt{cf} = \sqrt{df} - n$. Here we have exterminated the ſurd $\sqrt{b}$, and by each ſucceeding operation we take away one more, always adhering to the rules to transfer to the ſame ſide all thoſe ſurds whoſe ſuffixed quantities involve the ſame letter; from whence alſo it follows that the number of involutions will be plus one than the number of ſurds. In the ſecond ſtep there were eleven ſurds with rationals, and in the next tho' twelve appear;

 they

they can be reduced to three and a rational, whence the ſimplification is manifeſt: The ſame I have tried in a ſeptinomial, and the reſult was agreeable to the former; though the third involution, produced twenty-nine ſurd rectangles beſides rationals.

The example propoſed by Newton, where he mentions M. Fermat's method, will readily prove that the method juſt laid down ſhews the connection of the parts of the equation much better and is much more brief, the whole being carried on in one ſingle equation; that example is an equation conſiſting of a cubic ſurd, two quadratics and a rational, and which Sir Iſaac has given us without the operation. Doctor Hales in his Analyſis, 98th Section, has given the operation where even the ſubſtituted letters riſe to the cubic dimenſion. I ſhall now ſubjoin the operation according to the method now propoſed :

$\sqrt{ay} - \sqrt{a^2 - ay} - 2a = \sqrt[3]{ay^2}$ which cubed and collecting ſimilar terms becomes $-14a^3 + \overline{15a^2 - 2ay}\sqrt{ay} - \overline{13a^2 - 2ay}\sqrt{a^2 - ay} + 12a^2\sqrt{ay - y^2} = ay^2$; aſſuming $\overline{15a^2 - ay} = p$ and $\overline{13a^2 + 2ay} = q$ by tranſpoſition we have,

$\underline{p\sqrt{ay} + 12a^2\sqrt{ay - y^2} = ay^2 + 14a^3 + q\sqrt{a^2 - ay}}$ and ſquaring,

$p^2 ay + 144\overset{5}{a}y - 144\overset{4\ 2}{ay} + 24p\overset{2}{a}y\sqrt{a^2 - ay} = \overset{2\ 4}{ay} + 196\overset{6}{a} + \overset{2\ 2}{qa} - q\overset{2}{a}y$
$+ 28\overset{4\ 2}{ay} + \overline{2qay + 28\overset{3}{a}q}\sqrt{a^2 - ay}.$

whence

whence tranſpoſing and ſimplifying we have,

$$\overline{p^2+q^2}:ay+144\overset{5}{ay}-172\overset{4\,2}{ay}-196\overset{6}{a}-\overset{2\,4}{ay}-\overset{2\,2}{qa}=\begin{array}{l}\underline{+\,2qay}\\ \underline{+\,28\overset{3}{aq}}\\ \underline{-\,24\overset{2}{a}py}\end{array}\sqrt{a^2-ay}$$

which latter equation has but one ſurd, whence for $p.q.p^2$ and q^2 ſubſtituting their values and connecting the terms we have after dividing all by a^2

$$-\overset{4}{y}+8\overset{3}{ay}-184\overset{2\,2}{ay}+486\overset{3}{ay}-365\overset{4}{a}=\overline{74\overset{2}{ay}-304\overset{2}{ay}+364\overset{3}{a}+4\overset{3}{y}}$$

$:\sqrt{a^2\,ay}$ which equation ſquared and transformed into an original equation gives an equation free from ſurds the ſame as Doctor Hales's final equation.

$$\overset{8}{y}*+1008\overset{2\,6}{ay}-1464\overset{3\,5}{ay}-2762\overset{4\,4}{ay}+3680\overset{5\,3}{ay}+2916\overset{6\,2}{ay}-972\overset{7}{ay}+729\overset{8}{a}$$
$$=0.$$

By the ſame management we may take away the aſymmetry of an equation having ſurds of a more intricate nature.

Let there be propoſed the equation conſiſting of a cubic ſurd and two biquadratics $\sqrt[4]{a}+\sqrt[4]{b}=\sqrt[3]{c}$.

Involve both ſides to the cube and the equation will then ſtand,

$$\sqrt[4]{a^3}+3\sqrt[4]{ab^2}+3\sqrt[4]{a^2b}+\sqrt[4]{b^3}=c.$$

of theſe ſurds we may obſerve that the product of two pairs

 will

will give a fimple quadratic, placing at the fame fide thofe pairs the equation will ftand,

$\sqrt[4]{a^3} + 3\sqrt[4]{ab^2} = c - 3\sqrt[4]{a^2b} - \sqrt[4]{b^3}$ and fquaring this equation

$a\sqrt{a} + 6a\sqrt{b} + 9b\sqrt{a} = c + 9a\sqrt{b} + b\sqrt{b} - 6c\sqrt[4]{a^2b} - 2c\sqrt[4]{b^3} + 6b\sqrt{a}$

Let the terms be collected uniting fuch furds as admit it, and fubftituting letters for the coefficients $\overline{3a+b} = y$
$- 3b - a = z$

the equation will now be $6c\sqrt[4]{a^2b} + 2c\sqrt[4]{b^3} = c^2 + y\sqrt{b} - z\sqrt{a}$

whence by fquaring $36c^2a\sqrt{b} + 24c^2b\sqrt{a} + 4c^2b\sqrt{b} = c^4 + y^2b + z^2a + 2c^2y\sqrt{b} - 2c^2z\sqrt{a} - 2yz\sqrt{ab}.$

$$\text{Let}\quad 36c^2a + 4c^2b - 2c^2y = v$$
$$c^4 + y^2b + z^2a = x$$
$$24c^2b + 2c^2z = t$$

The equation will then by tranfpofition ftand thus,

$v\sqrt{b} + t\sqrt{a} = x - 2yz\sqrt{ba}$ and fquaring

$v^2b + t^2a + 2vt\sqrt{ab} = x^2 - 4yzx\sqrt{ba} + 4y^2z^2ba$ let the rationals $= r$ and the coefficient $2vt + 4yzx = s$ and the equation will be $s\sqrt{ab} = r$ and fquaring $s^2ab = r^2$ where all the terms are rational.

NEARLY

NEARLY ſimilar is the management of three biquadratic ſurds, by all which it will appear that this method is much ſhorter than M. Fermat's; the principal care to be taken is to keep at the ſame ſide two ſurds whoſe product will give a ſimple quadratic, and then uniting the terms that will admit of it proceed as before by involution and tranſpoſition.

WHEN fractions having ſurds in their denominators occur, it is expedient to remove the ſurds out of the denominator by multiplication, this is uſually done by the multiplication of the denominator taken as a binomial or reſidual; if there be four quadratic ſurds this is generally ſuppoſed the limit, and that if a fifth term be added whether rational or not, the denominator cannot be rendered rational, but ſince the binomial or reſidual here affords the ſame convenience as the tranſpoſition in equations, a repeated multiplication will clear the denominator of radicality. True it is that after multiplication by a binomial, if the denominator be reſolved into parts of one and four ſurds, there will come out ſix ſurds and in no caſe leſs than four and a rational, however by continuing the operation a little farther the ſimplification will appear,

$$\frac{m, \&c.}{\overline{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{a}+\sqrt{e}}} \times \overline{\sqrt{a}+\sqrt{b}-\sqrt{c}-\sqrt{d}-\sqrt{e}}$$ the denominator which is the multiplier being reſolved into parts of two and

and three furds, in which cafe fewer furds will come out in the product after uniting fuch as admit of it, than if the denominator were refolved into parts of four and one, by retaining at the fame fide of the binomial thofe furds whofe quantities under the radical fign have a common divifor, in the next multiplication their product coalefce and may be united, fo that after four multiplications the denominator is freed from radicality as will beft appear by example,

$$\overline{\surd a + \surd b + \surd c + \surd d + \surd e} \times \overline{\surd a + \surd b - \surd c - \surd d - \surd e}$$

$$\begin{array}{l} +a \quad +\surd ab + \surd ac + \surd ad + \surd ae + \surd bc + \surd bd + \surd be - \surd cd - \surd ec - \surd ed \\ +b \quad +\surd ab - \surd ac - \surd ad - \surd ae - \surd bc - \surd bd - \surd be - \surd cd - \surd ec - \surd ed \\ -c \\ -d \\ -e \\ \hline a + b - c - d - e + 2\surd ab \quad * \; * \; * \; * \; * \; * \quad - 2\surd cd - 2\surd ce \; 2\surd cd \end{array}$$

Let all the rationals equal r and the product will be

$$r + 2\surd ab - 2\surd ce - 2\surd cd - 2\surd de$$

$$r + 2\surd ab - 2\surd ce - 2\surd cd - 2\surd ed \times r + 2\surd ab - \surd ce + 2\surd cd + 2\surd de$$

$$\begin{array}{l} r^2 + 2r\surd ab - 2r\surd ce - 2r\surd cd - 2r\surd de - 4\surd\overline{abce} - 4\surd\overline{abcd} - 4\surd\overline{abde} \\ 4ab + 2r\surd ab - 2r\surd ce + 4e\surd cd + 4c\surd de - 4\surd\overline{abce} + 4\surd\overline{abcd} + 4\surd\overline{abcde} \\ 4ce \qquad - 4d\surd ce + 2r\surd cd - 4c\surd de \\ 4cd \qquad - 4d\surd ce - 4e\surd cd + 2r\surd de \\ 4de \\ \hline r^2 + 4ab + 4ce - 4cd - 4de + 4r\surd ab - \overline{4r - 8d}\surd ce - 8\surd\overline{abce} \end{array}$$

Let

LET all the rationals $= p$ and the coefficient $-4r-8d=q$ and the product will now be $p+4r\sqrt{ab}-q\sqrt{ce}-8\sqrt{abce}$ by observing the same rules and multiplying by $p+4r\sqrt{ab}+q\sqrt{ce}+8\sqrt{abce}$ the product will have but one surd.

$$\overline{p+4r\sqrt{ab}-q\sqrt{ce}-8\sqrt{abce}}\times\overline{p+4r\sqrt{ab}+q\sqrt{ce}+8\sqrt{abce}}$$

will give $p^2+16r^2ab-q^2ce-64abce+\overline{8rp-16qce}\sqrt{ab}$ now 'tis plain that by one multiplication more the denominator is free from all radicals.